Background (p.27 in [1]): A first order differential equation $y' = f(x,y)$ can be generalized by

$$M(x,y) + N(x,y)y' = 0 \ . \tag{1}$$

Let

$$\begin{aligned}\frac{\partial \varphi}{\partial x} &= M(x,y) \\ \frac{\partial \varphi}{\partial y} &= N(x,y) \ .\end{aligned} \tag{2}$$

Now, (1) can be expressed as

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0 \ , \tag{3}$$

which, by the chain rule, is the same as

$$\frac{d}{dx}\varphi(x, y(x)) = 0 \ . \tag{4}$$

But this means that

$$\varphi(x, y(x)) = C \ , \tag{5}$$

with C constant is the solution.

Problem to solve: On page 208, we now desire to solve

$$c[y(b) - b]y'(b) - y(b) \quad = \quad 0 \ ,$$

where $c = (N-1)$. Let

$$M(b,y) \quad = \quad -y(b) \ ; \tag{6}$$

$$N(b,y) \quad = \quad c(y(b) - b) \ , \tag{7}$$

Taking the partial differentiation of (12) and (13), we can have

$$\frac{\partial M}{\partial y} \quad = \quad -1 \ ; \tag{8}$$

$$\frac{\partial N}{\partial b} \quad = \quad -c \ . \tag{9}$$

According to the definition of the exactness of Thm. 1.1 in [1], we find that $(\partial M)/(\partial y) \neq (\partial N)/(\partial b)$. Thus, we need to find an integrating factor μ such that

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial b} \ . \tag{10}$$

Multiplying (6) by the integrating factor μ, we obtain

$$c(y-b)\mu y' - \mu y = 0 \tag{11}$$

and

$$M(b,y) \quad = \quad -\mu y \ ; \tag{12}$$

$$N(b,y) \quad = \quad c(y-b)\mu \ , \tag{13}$$

By the text for exactness, i.e. $(\partial M)/(\partial y) = (\partial N)/(\partial b)$, we now need a μ such that

$$-\frac{\partial \mu}{\partial y} y - \mu = c(y-b)\frac{\partial \mu}{\partial b} - c\mu \ . \tag{14}$$

Let $(\partial \mu)/(\partial b) = 0$ for simplicity, we now have

$$\begin{aligned} -\frac{\partial \mu}{\partial y} y - \mu &= -c\mu \\ \frac{\partial \mu}{\partial y} y &= (c-1)\mu \\ \frac{\partial \mu}{\mu} &= (c-1)\frac{\partial y}{y} \ . \end{aligned} \tag{15}$$

Integrating (15) with respect to μ and y gives

$$\begin{aligned} \ln \mu &= (c-1)\ln y \\ \mu &= y^{c-1} \ . \end{aligned} \tag{16}$$

Now, the $M(b,y)$ and $N(b,y)$ can be expressed as

$$M(b,y) = -y^c \ ; \tag{17}$$

$$N(b,y) = -c(y-b)y^{c-1} \ . \tag{18}$$

Next, we should find a φ such that

$$\frac{\partial \varphi}{\partial b} = M = -y^c \ ; \tag{19}$$

$$\frac{\partial \varphi}{\partial y} = N = c(y-b)y^{c-1} \ . \tag{20}$$

From $(\partial \varphi)/(\partial b) = M$, we obtain

$$\varphi = -by^c + g(y) \ . \tag{21}$$

Substituting (21) into (20), we get

$$\begin{aligned} \frac{\partial \varphi}{\partial y} &= -cby^{c-1} + g'(y) \\ &= c(y-b)y^{c-1} \\ &= cy^c - cby^{c-1} \ . \end{aligned} \tag{22}$$

Thus, we get $g'(y) = cy^c$, which means $g(y) = (c)/(c+1)y^{c+1}$. Then, φ can be written as

$$\varphi = -by^c + \frac{c}{c+1}y^{c+1} \ . \tag{23}$$

Recall that the solution to the first order differential equation (6) is $\varphi = d$, where d is a constant. With $y(0) = 0$, we have $d = 0$.

At last, the solution can be written as

$$\begin{aligned} by^c &= \frac{c}{c+1}y^{c+1} \\ y &= = \frac{c+1}{c}b \ \ (c = N-1) \\ y &= \frac{N}{N-1}b \ . \end{aligned} \tag{24}$$

[1]: Peter V. O'Neil, *Advanced Engineering Mathematics*, international student edition, Thomson, 2007.